Choosing $C^{(\pm)}_{jm}$ to be real and positive,

$$J_\pm |j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,|j,m\pm 1\rangle .$$

(You can check this for J_- similarly.)

$$\Rightarrow \quad \langle j',m'|J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\cdot\hbar\cdot\delta_{j'j}\,\delta_{m',m\pm 1}$$

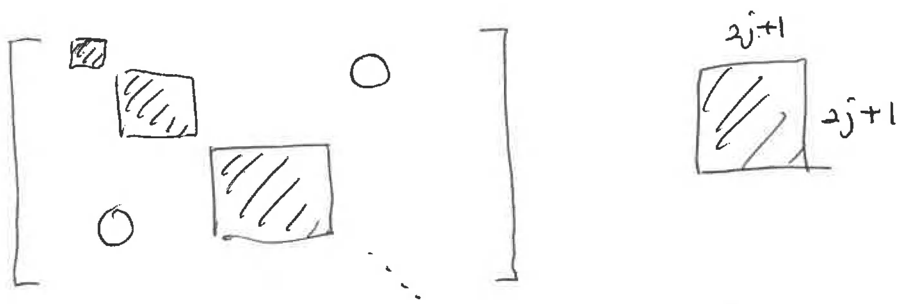
- Representations of the Rotation Operator

$$\mathcal{D}^{(j)}_{m'm}(R) \equiv \langle j,m'|\exp\left[-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi\right]|j,m\rangle$$

(Wigner function) or d-matrix : a matrix element of $\mathcal{D}(R)$

NOTE : it's diagonal in $|j\rangle$. $\parallel$ $\vec{J}|j\rangle \propto |j\rangle$ //

$\hookrightarrow$ a block-diagonal matrix



$(m = -j \cdots j)$

– $(2j+1)$ by $(2j+1)$ –

$\Rightarrow$ The rotation matrices characterized by definite j : $\boxed{/\!/}$

form a "group". $\parallel$ NOTE : 2 is the dimension of the "defining, fundamental" one on SU(2).

- Identity : $\phi = 0$.
- Inverse : $\phi \to -\phi$
- Composition :

$$\sum_{m'} \mathcal{D}^{(j)}_{m''m'}(R_1)\,\mathcal{D}^{(j)}_{m'm}(R_2) = \mathcal{D}^{(j)}_{m''m}(R_1R_2) .$$ //

- unitarity : $\mathcal{D}_{m'm}(R^{-1}) = \mathcal{D}^*_{mm'}(R)$.

• Rotation of $|j,m\rangle \longrightarrow \mathcal{D}(R)|j,m\rangle$

$$\Rightarrow \mathcal{D}(R)|jm\rangle = \underbrace{\sum_{m'} |jm'\rangle\langle jm'|}_{\text{completeness}}\mathcal{D}(R)|jm\rangle$$

$$= \sum_{m'} |jm'\rangle \mathcal{D}^{(j)}_{m'm}(R)$$

There are different ways of computing $\mathcal{D}^{(j)}_{m'm}(R)$:

Sakurai introduces two.

① a direct method for low j : $(j=\frac{1}{2}, 1)$

consider the realization with Euler angles,

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle jm'| e^{-\frac{i}{\hbar}J_z\alpha} e^{-\frac{i}{\hbar}J_y\beta} e^{-\frac{i}{\hbar}J_z\gamma} |jm\rangle$$

$$= e^{-i(m'\alpha+m\gamma)} \langle jm'| e^{-\frac{i}{\hbar}J_y\beta} |jm\rangle.$$

Wigner "small" d-matrix $\equiv d^{(j)}_{m'm}(\beta)$

$\hookrightarrow d^{(j)}_{m'm}(\beta) = \langle jm'| e^{-\frac{i}{\hbar}J_y\beta} |jm\rangle$

Can be directly computed for $j=\frac{1}{2}$ and $j=1$.

a. $j=\frac{1}{2}$: We know how to do this

$$\Rightarrow \quad d^{(\frac{1}{2})} = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

We know :

$$\left(\; U(\beta,\hat{n}) = \cos\frac{\beta}{2}\cdot I - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\beta}{2} \; \right).$$

b. $j=1$: use $J_y = \frac{1}{2i}(J_+ - J_-)$

$$\Rightarrow J_y = \frac{\hbar}{2}\begin{array}{cc} \begin{pmatrix} 0 & -\sqrt{2}\,i & 0 \\ \sqrt{2}\,i & 0 & -\sqrt{2}\,i \\ 0 & \sqrt{2}\,i & 0 \end{pmatrix} & \begin{array}{l} m'=1 \\ m'=0 \\ m'=-1 \end{array} \\ \begin{array}{ccc} m=1 & m=0 & m=-1 \end{array} & \end{array}$$

$\Leftarrow \langle j'm'|J_\pm|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,\delta_{jj'}\,\delta_{m',m\pm1}$

$\hookrightarrow$ This matrix has a property. $\left(\frac{J_y}{\hbar}\right)^3 = \frac{J_y}{\hbar}$, but $\left(\frac{J_y}{\hbar}\right)^2 \neq 1$.

$$\Rightarrow \exp\left(-\frac{i}{\hbar}J_y\beta\right) = 1 - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta) - i\left(\frac{J_y}{\hbar}\right)\sin\beta$$

* for $j=1$ only !

verification

$$e^{-\frac{i}{\hbar}J_y\beta} = 1 - i\beta\frac{J_y}{\hbar} - \frac{1}{2}\beta^2\left(\frac{J_y}{\hbar}\right)^2 + \frac{(-i\beta)^3}{3!}\frac{J_y}{\hbar} + \frac{\beta^4}{4!}\left(\frac{J_y}{\hbar}\right)^2 + \cdots$$

$$= 1 - \left(\frac{J_y}{\hbar}\right)^2 + \left(\frac{J_y}{\hbar}\right)^2\underbrace{[\text{ even terms }]}_{\cos\beta} - i\frac{J_y}{\hbar}\underbrace{[\text{ odd terms }]}_{\sin\beta}$$

$$\Rightarrow d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

BAD : it's not systematic at all.

it's not possible to do this for a high j.

② Schwinger's oscillator Model [Ch. 3.9, S&N]

$$\Rightarrow \quad d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k-m+m'} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(j-k-m')!\,(k-m+m')!} \cdot \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \cdot \left(\sin\frac{\beta}{2}\right)^{2k-m+m'}$$

"Wigner d-matrix"

Symmetry properties

- $d^{(j)}_{pq}(\beta) = (-1)^{p-q}\, d^{(j)}_{-p,-q}(\beta) = d^{(j)}_{-q,-p}(\beta)$
- $d^{(j)}_{pq}(-\beta) = (-1)^{p-q}\, d^{(j)}_{pq}(\beta) = d^{(j)}_{qp}(\beta)$
- $d^{(j)}_{pq}(\pi-\beta) = (-1)^{j-q}\, d^{(j)}_{-pq}(\beta) = (-1)^{j+p}\, d^{(j)}_{p,-q}(\beta)$
- $\boxed{d^{(j)}_{pq}(\beta \pm 2\pi n) = (-1)^{2jn}\, d^{(j)}_{pq}(\beta)}$ ☆☆☆
- $d^{(j)}_{pq}(\beta \pm (2n+1)\pi) = (-1)^{\pm(2n+1)j-q}\, d^{(j)}_{p,-q}(\beta)$

$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$: "4π"-periodicity! ☆☆☆

$j = 0, 1, 2, \ldots$: 2π-periodicity.

Schwinger: $[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$ can be implemented by using two uncoupled oscillators, or "bosons"!

oscillator ↑ : $(a_\uparrow^\dagger, a_\uparrow) \to |n_\uparrow\rangle \to n_\uparrow$ ↑-spins carrying $|\frac{1}{2}, \frac{1}{2}\rangle$

oscillator ↓ : $(a_\downarrow^\dagger, a_\downarrow) \to |n_\downarrow\rangle \to n_\downarrow$ ↓-spins carrying $|\frac{1}{2}, -\frac{1}{2}\rangle$

$\to$ $|j, m\rangle$ can be represented by $(j+m)$ ↑-spins and $(j-m)$ ↓-spins.

$\Rightarrow |j,m\rangle \propto (\tilde{a}_\uparrow^+)^{j+m} (\tilde{a}_\downarrow^+)^{j-m} |0\rangle$

: It's like the addition of $2j$ spin-$\frac{1}{2}$ particles.

(Schwinger boson representation)

- Rotation Matrix

$$\mathcal{D}(R)|jm\rangle \propto \mathcal{D}(R)(\tilde{a}_\uparrow^+)^{j+m}(\tilde{a}_\downarrow)^{j-m}|0\rangle$$

$$= \underbrace{\mathcal{D}\tilde{a}_\uparrow^+\mathcal{D}^{-1}\mathcal{D}\tilde{a}_\uparrow^+\cdots\tilde{a}_\uparrow^+\mathcal{D}^{-1}}_{j+m}\underbrace{\mathcal{D}\tilde{a}_\downarrow^+\mathcal{D}^{-1}\mathcal{D}\cdots\mathcal{D}\tilde{a}_\downarrow^+\mathcal{D}^{-1}}_{j-m}\mathcal{D}|0\rangle$$

$$= (\mathcal{D}(R)\tilde{a}_\uparrow^+\mathcal{D}^{-1}(R))^{j+m}(\mathcal{D}(R)\tilde{a}_\downarrow^+\mathcal{D}^{-1}(R))^{j-m}\underbrace{|0\rangle}_{=\mathcal{D}|0\rangle}$$

☆. Thus, the rotation matrix of spin-j is determined by the rotation of spin-$\frac{1}{2}$ operators.

In detailed calculations,

we define $J_+ \equiv \hbar\tilde{a}_\uparrow^+\tilde{a}_\downarrow$, $J_- \equiv \hbar\tilde{a}_\downarrow^+\tilde{a}_\uparrow$

m-raising m-lowering

removing $\downarrow$-spin
adding $\uparrow$-spin

removing $\uparrow$-spin
adding $\downarrow$-spin

$$J_z \equiv \frac{\hbar}{2}(\tilde{a}_\uparrow^+\tilde{a}_\uparrow - \tilde{a}_\downarrow^+\tilde{a}_\downarrow) = \frac{\hbar}{2}(\tilde{N}_\uparrow - \tilde{N}_\downarrow)$$

$\Rightarrow$ satisfying all commutation relations of $\vec{J}$.

Thus, $\boxed{j \equiv \frac{n_\uparrow + n_\downarrow}{2}}$, $\boxed{m \equiv \frac{n_\uparrow - n_\downarrow}{2}}$

$$J^2 = \frac{\hbar^2}{2}\tilde{N}\cdot\left(\frac{\tilde{N}}{2}+1\right) \quad // \quad \tilde{N} = \tilde{N}_\uparrow + \tilde{N}_\downarrow$$

and,

$$|n_\uparrow, n_\downarrow\rangle = \frac{(a_\uparrow^+)^{n_\uparrow}(a_\downarrow^+)^{n_\downarrow}}{\sqrt{n_\uparrow!}\ \sqrt{n_\downarrow!}}|0,0\rangle$$

$$\rightarrow\quad |j,m\rangle = \frac{(a_\uparrow^+)^{j+m}(a_\downarrow)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle.$$

Then, the rotation matrix is written as

$$\circ\ \mathcal{D}(R)|j,m\rangle = \frac{[\mathcal{D}(R)\,a_\uparrow^+\,\mathcal{D}^{-1}(R)]^{j+m}\,[\mathcal{D}(R)\,a_\downarrow^+\,\mathcal{D}^{-1}(R)]^{j-m}}{\sqrt{(j+m)!\ \ (j-m)!}}|0\rangle \quad \text{---- } (*)$$

choose $\alpha=0$, $\gamma=0$ for the Euler angles to produce the Wigner "small" d-matrix :

Compare!

$$\circ\ \mathcal{D}(\alpha=0,\beta,\gamma=0)|j,m\rangle = \sum_{m'}|j\,m'\rangle\, d^{(j)}_{m'm}(\beta)$$

$$= \sum_{m'} d^{(j)}_{m'm}(\beta)\,\frac{(a_\uparrow^+)^{j+m'}(a_\downarrow^+)^{j-m'}}{\sqrt{(j+m')!\,(j-m')!}}|0\rangle \quad \text{---- } (**)$$

↳ Rotation of $a_\uparrow^+, a_\downarrow^+$: $(\alpha=0, \gamma=0) \rightarrow \mathcal{D}(R) = e^{-\frac{i}{\hbar}J_y\beta}$

$$\mathcal{D}(R)\,a_\uparrow^+\,\mathcal{D}^{-1}(R) = a_\uparrow^+\cos\frac{\beta}{2} + a_\downarrow^+\sin\frac{\beta}{2}$$

$$\mathcal{D}(R)\,a_\downarrow^+\,\mathcal{D}^{-1}(R) = -a_\uparrow^+\sin\frac{\beta}{2} + a_\downarrow^+\cos\frac{\beta}{2}.$$

$$\left[-\frac{J_y}{\hbar}, a_\uparrow^+\right] = \frac{a_\downarrow^+}{2i}$$

$$\left[-\frac{J_y}{\hbar}, a_\downarrow^+\right] = \frac{i}{2}a_\uparrow^+$$

↳ using the binomial theorem : $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$.

We can expand $(*)$ in terms of $(a_\uparrow^+)^p(a_\downarrow^+)^q$.

↳ Comparing with $(**)$,

$$d^{(j)}_{m'm}(\beta) = \sum_{k}(-1)^{k-m+m'}\frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(j-k-m')!\,(k-m+m')!}\cdot\left[\cos\frac{\beta}{2}\right]^{2j-2k+m-m'}\left[\sin\frac{\beta}{2}\right]^{2k-m+m'}$$

Sum runs over k that doesn't make ( )! negative.